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ON LITTLEWOOD'S BOUNDEDNESS PROBLEM FOR SUBLINEAR DUFFING EQUATIONS

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Abstract. In this paper, we are concerned with the boundedness of *all* the solutions and the existence of quasi-periodic solutions for second order differential equations

$$x'' + g(x) = e(t),$$

where the 1-periodic function $e(t)$ is a smooth function and $g(x)$ satisfies sublinearity:

$$\mathrm{sign}(x) \cdot g(x) \to +\infty, \quad g(x)/x \to 0 \quad \text{as } |x| \to +\infty.$$

1. Introduction

In this paper, we will investigate the boundedness of *all* solutions of the conservative system

$$x'' + g(x) = e(t), \quad (\,' = \frac{d}{dt} \,) \tag{1.1}$$

where $e(t)$ is 1-periodic in t.

As one of the simplest but non-trivial conservative systems, Eq. (1.1) has been widely studied for a long time. For example, many authors studied the existence and multiplicity of periodic solutions by various methods, such as, critical point theory, phase-plane analysis combined with the shooting methods or fixed point theorems of planar homeomorphisms and continuation methods based on degree theory.

In the early 60's, Littlewood [6] proposed to study the boundedness of all the solutions of (1.1) in the following two cases:

(1) Superlinear case: $g(x)/x \to +\infty$ as $x \to \pm\infty$;

(2) Sublinear case: $sign(x) \cdot g(x) \to +\infty$ and $g(x)/x \to 0$ as $x \to \pm\infty$.

The first result in the superlinear case is due to Morris [11], who proved that all solutions of

$$x'' + 2x^3 = e(t)$$

are bounded, where $e(t) \in \mathcal{C}^0$. Subsequently, this result was extended to the more general case by several authors, we refer to [2], [4], [5], [7], [16] and references therein.

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Recently, the boundedness of solutions for the following simple equation

$$x'' + |x|^{\alpha-1}x = e(t),$$

has been studied in [3] and [8], where $0 < \alpha < 1$. They proved that every solution is bounded if $e(t) = e(t+1)$ is a smooth function. As far as we know, this is only one example in sublinear case.

In what follows, we denote by $G(x)$ the integral of $g(x)$ with $G(0) = 0$. That is,

$$G(x) = \int_0^x g(s)ds. \tag{1.2}$$

We also denote by $c < 1$ and $C > 1$, respectively, two universal positive constants not concerning their quantities.

The main result of this paper is the following

Theorem 1. *Assume that $e(t) \in \mathcal{C}^5$ is 1-periodic in t, the smooth function $g(x) \in \mathcal{C}^6(\mathbb{R})$ satisfies that for all $x \neq 0$,*

(i) $xg(x) > 0$ and there are two positive constants σ_1, σ_2

$$\sigma_1 \leq \frac{G(x)G''(x)}{g^2(x)} \leq \frac{1}{2} - \sigma_2; \tag{1.3}$$

(ii)

$$\left| x^k \frac{d^k}{dx^k} G(x) \right| \leq C \cdot G(x), \quad \text{for } k \leq 7. \tag{1.4}$$

(iii) $c \cdot G(x) \leq G(-x) \leq C \cdot G(x)$.

Then every solution of (1.1) is bounded, i.e., if $x = x(t)$ is a solution of (1.1), then it is defined in $(-\infty, +\infty)$ and

$$\sup_{t \in \mathbb{R}} (|x(t)| + |x'(t)|) < +\infty.$$

Remark 1. The assumptions on $g(x)$ in (i), (ii) and (iii) can be weakened to require that they hold for $|x| \geq d$ for any fixed constant $d > 0$.

Remark 2. From the conditions (i) and (ii), it is easy to see that

$$0 \leq g'(x) \leq C, \quad g'(x) \to 0 \quad \text{as } |x| \to \infty, \tag{1.5}$$

$$\operatorname{sign}(x) \cdot g(x) \to +\infty, \quad G(x)/x^2 \to 0 \quad \text{as } |x| \to +\infty, \tag{1.6}$$

$$xg(x) \geq G(x), \quad x^2 g'(x) \geq c \cdot G(x). \tag{1.7}$$

Remark 3. The inequalities in (1.3) and (1.4) are similar to those in [5, (1.2), (1.3)]. We would like to point out that there are also some delicate estimates in [5], which will be used in our proof.

Example. Every solution of the equation

$$x'' + \frac{x}{(1+x^2)^{1/3}} = \cos 2\pi t$$

is bounded.

The rest of the paper is organized as follows. Some technical lemmas which are useful for our proof are stated in §2 and §3. We will give the proof of Theorem 1 in §4 and another theorem about the existence of quasi-periodic solutions, Aubry-Mather set and unlinked periodic solutions.

The idea for proving the boundedness of solutions of Eq. (1.1) is as follows. By means of transformation theory, (1.1) is, outside of a large disc $\mathcal{D} = \{(x, x') \in \mathbb{R}^2 : x^2 + x'^2 \leq r^2\}$ in the (x, x')-plane, transformed into a perturbation of an integrable Hamiltonian system. The Poincaré map of the transformed system is close to a so-called twist map in $\mathbb{R}^2 \backslash \mathcal{D}$. Then Moser's twist theorem [13] guarantees the existence of arbitrarily large invariant curves diffeomorphic to circles and surrounding the origin in the (x, x')-plane. Every such curve is the base of a time-periodic and flow-invariant cylinder in the extended phase space $(x, x', t) \in \mathbb{R}^2 \times \mathbb{R}$, which confines the solutions in the interior and which leads to a bound of these solutions.

2. Action-angle variables and some estimates

In this section, we first introduce action-angle variables for Eq. (1.1) and then state some technical lemmas which will be used in the proofs of our main result.

Because of Remark 1, without loss of generality and for brevity of arguments, we assume that the average value of $e(t)$ vanishes, i.e.,

$$\int_0^1 e(t)dt = 0.$$[1]

Hence the function

$$E(t) = \int_0^t e(s)ds$$

is also 1-periodic in t and is in $\mathcal{C}^6$.

Eq. (1.1) is equivalent to the planar Hamiltonian system

$$x' = \frac{\partial h}{\partial y}(x, y, t), \quad y' = -\frac{\partial h}{\partial x}(x, y, t), \tag{2.1}$$

where

$$h(x, y, t) = \frac{1}{2}y^2 + G(x) + yE(t).$$

Consider an auxiliary autonomous system

$$x' = y, \quad y' = -g(x), \tag{2.2}$$

which is an integrable Hamiltonian system with Hamiltonian function

$$\mathcal{H}(x, y) = \frac{1}{2}y^2 + G(x).$$

The closed curves $\mathcal{H}(x, y) = h > 0$ are just the integral curves of (2.2). It is well known from [1] that (2.2) has action and angle variables (I, θ). Now we give an explicit form of them.

Denote by $T(h)$ the time period of the integral curve Γ_h of (2.2) defined by $\mathcal{H}(x, y) = h$ and by I, the area surrounded by the closed curve Γ_h. Then

$$I = \oint_{\mathcal{H}(x,y)=h} ydx := J_0(h).$$

[1]In fact, if $\int_0^1 e(t)dt \neq 0$, we can use $\tilde{g}(x) = g(x) - \int_0^1 e(t)dt$ instead of $g(x)$. It is easy to verify that $\tilde{g}(x)$ satisfies all the conditions of Theorem 1 for any $|x| \geq d$ with some positive constant d.

Denote by G_+^{-1} and G_-^{-1} the right and left inverse of G, respectively. Assume $(x_+, 0)$ and $(x_-, 0)$ are the intersection points of Γ_h with x-axis, i.e.,

$$x_- = G_-^{-1}(h) < 0 < G_+^{-1}(h) = x_+.$$

By the symmetry condition (iii) of Theorem 1, we have

$$1 \le \frac{\max\{G_+^{-1}(h), |G_-^{-1}(h)|\}}{\min\{G_+^{-1}(h), |G_-^{-1}(h)|\}} \le C. \tag{2.3}$$

It is easy to see that

$$J_0(h) = 2\int_{x_-}^{x_+} \sqrt{2(h - G(s))}ds, \quad T(h) = 2\int_{x_-}^{x_+} \frac{1}{\sqrt{2(h - G(s))}}ds. \tag{2.4}$$

For any $(x, y) \in \mathbb{R}^2$, set $h = \frac{1}{2}y^2 + G(x)$. We define the generating function $S(x, I)$,

$$S(x, I) = \int_{\mathcal{C}} ydx,$$

where $\mathcal{C}$ is the part of the level curve $\Gamma_h : \mathcal{H}(x, y) = h$ connecting the y-axis with the point (x, y), oriented clockwise and $h = J_0^{-1}(I)$. This defines S up to an integer multiple of I, the area of the domain surrounded by the closed curve Γ_h. We define the map $(\theta, I) \mapsto (x, y)$ by

$$y = \frac{\partial S}{\partial x}(x, I), \quad \theta = \frac{\partial S}{\partial I}(x, I), \tag{2.5}$$

which is well-known to be symplectic because

$$dx \wedge dy = S_{xI} dx \wedge dI = d\theta \wedge dI.$$

Eq. (2.2) in the new coordinates (θ, I) is also a Hamiltonian system with the new Hamiltonian function

$$h_0(\theta, I) = J_0^{-1}(I).$$

Under this symplectic map, the system (2.1) becomes

$$I' = -\frac{\partial \varrho}{\partial \theta}(\theta, I, t), \quad \theta' = \frac{\partial \varrho}{\partial I}(\theta, I, t), \tag{2.6}$$

where

$$\varrho(\theta, I, t) = J_0^{-1}(I) + y(\theta, I)E(t) := J_0^{-1}(I) + H_1(\theta, I, t) \tag{2.7}$$

with $(x, y) = (x(\theta, I), y(\theta, I))$ defined implicitly by (2.5).

In the following, we state some lemmas which will be used in §3 and §4.

Lemma 2.1. *The following inequalities hold:*

$$\sqrt{2h} \cdot (G_+^{-1}(h) + |G_-^{-1}(h)|) \le J_0(h) \le 2\sqrt{2h} \cdot (G_+^{-1}(h) + |G_-^{-1}(h)|), \tag{2.8}$$

$$c \cdot h^{-1} \cdot J_0(h) \le T(h) = J_0'(h) \le C \cdot h^{-1} \cdot J_0(h), \tag{2.9}$$

$$c \cdot h^{-1} J_0'(h) \le T'(h) = J_0''(h) \le C \cdot h^{-1} J_0'(h). \tag{2.10}$$

Moreover, $J_0'(h) \to +\infty$ *as* $h \to +\infty$.

Proof. (1) One can prove the inequalities in (2.8) by comparing the area bounded by Γ_h respectively with the area of the triangle or rectangle with sides $\sqrt{2h}$ and $|G_{\pm}^{-1}(h)|$.

(2) From the definitions of $J_0(h)$ and $T(h)$, it follows that

$$J_0'(h) = T(h).$$

On the other hand, we have

$$J_0'(h) = \frac{2}{h}\int_{x_-}^{x_+}\left(\frac{3}{2} - \frac{G(\xi)G''(\xi)}{g^2(\xi)}\right)\sqrt{2(h-G(\xi))}d\xi. \tag{2.11}$$

In fact, let

$$J_0^+(h) = 2\int_0^{x_+}\sqrt{2(h-G(\xi))}d\xi, \quad J_0^-(h) = 2\int_{x_-}^0\sqrt{2(h-G(\xi))}d\xi.$$

Then

$$J_0(h) = J_0^+(h) + J_0^-(h), \quad J_0'(h) = \frac{d}{dh}J_0^+(h) + \frac{d}{dh}J_0^-(h).$$

Let $G(\xi) = sh$. Then $hd\xi/dh = G(\xi)/g(\xi)$ and

$$J_0^+(h) = 2\int_0^1 \frac{h\eta}{g(\xi)}ds$$

where $\eta = \sqrt{2(1-s)h}$. So we obtain

$$\begin{aligned}
\frac{d}{dh}J_0^+(h) &= 2\int_0^1 \frac{\eta g(\xi) + \frac{\eta g(\xi)}{2} - \eta G''(\xi)\frac{G(\xi)}{g(\xi)}}{g^2(\xi)}ds \\
&= \frac{2}{h}\int_0^1\left(\frac{3}{2} - \frac{G(\xi)G''(\xi)}{g^2(\xi)}\right)\cdot\frac{h\eta}{g(\xi)}ds \\
&= \frac{2}{h}\int_0^{x_+}\left(\frac{3}{2} - \frac{G(\xi)G''(\xi)}{g^2(\xi)}\right)\cdot\sqrt{2(h-G(\xi))}d\xi.
\end{aligned}$$

Similarly, one can prove

$$\frac{d}{dh}J_0^-(h) = \frac{2}{h}\int_{x_-}^0\left(\frac{3}{2} - \frac{G(\xi)G''(\xi)}{g^2(\xi)}\right)\cdot\sqrt{2(h-G(\xi))}d\xi.$$

Hence

$$J_0'(h) = \frac{2}{h}\int_{x_-}^{x_+}\left(\frac{3}{2} - \frac{G(\xi)G''(\xi)}{g^2(\xi)}\right)\cdot\sqrt{2(h-G(\xi))}d\xi.$$

By the condition (i) of Theorem 1, we have

$$(\frac{3}{2} - \sigma_1)\frac{1}{h}\cdot J_0(h) \geq T(h) = J_0'(h) \geq (1+\sigma_2)\frac{1}{h}\cdot J_0(h), \tag{2.12}$$

which yields that

$$C\cdot h^{\frac{3}{2}-\sigma_1} \geq J_0(h) \geq c\cdot h^{1+\sigma_2}. \tag{2.13}$$

Hence,

$$C\cdot h^{-1}\cdot J_0(h) \geq T(h) = J_0'(h) \geq c\cdot h^{-1}\cdot J_0(h) \geq c\cdot h^{\sigma_2} \to \infty$$

as $h \to +\infty$.

Using the same trick in computing $J_0'(h)$, one can see that

$$J_0''(h) = T'(h) = \frac{2}{h}\int_{x_-}^{x_+}\left(\frac{1}{2} - \frac{G(\xi)G''(\xi)}{g^2(\xi)}\right)\cdot\frac{1}{\sqrt{2(h-G(\xi))}}d\xi.$$

Hence, by the condition (i) in Theorem 1, we have

$$c\cdot h^{-1}J_0'(h) = c\cdot h^{-1}T(h) \le J_0''(h) \le C\cdot h^{-1}T(h) = C\cdot h^{-1}J_0'(h).$$

The proof is finished. □

Lemma 2.2. *For $k \le 7$, we have*

$$\left|\frac{d^kJ_0}{dh^k}(h)\right| \le C\cdot h^{-k}J_0(h), \tag{2.14}$$

$$\left|\frac{\partial^k x}{\partial I^k}(\theta,I)\right| \le C\cdot I^{-k}|x|, \quad \left|\frac{\partial^k y}{\partial I^k}(\theta,I)\right| \le C\cdot I^{-k}|y|. \tag{2.15}$$

The proof can be found in [5][2].

By (2.9) and (2.14), one can verify that

(2.16)
$$c\cdot I^{-1}J_0^{-1}(I) \le \frac{d}{dI}J_0^{-1}(I) \le C\cdot I^{-1}J_0^{-1}(I), \quad \left|\frac{d^k}{dI^k}T(h)\right| \le C\cdot I^{-k}T(h).$$

From the definition of θ, it follows that

$$\frac{\partial x}{\partial\theta}(\theta,I) = T(h)y, \quad \frac{\partial y}{\partial\theta}(\theta,I) = -T(h)g(x). \tag{2.17}$$

Lemma 2.3.

$$\left|\frac{\partial^{k+i+\ell}}{\partial I^k\partial\theta^i\partial t^\ell}H_1(\theta,I,t)\right| \le C\cdot I^{-k}\cdot\sqrt{J_0^{-1}(I)} \tag{2.18}$$

for $k+i\le 7$, $i=0,1$ and $\ell\le 6$.

Proof. From the definition of $H_1(\theta,I,t)$, we have

$$\frac{\partial^{k+\ell}}{\partial I^k\partial t^\ell}H_1(\theta,I,t) = \frac{\partial^k y}{\partial I^k}\cdot\frac{d^\ell}{dt^\ell}E(t).$$

The conclusion for $i=0$ follows easily from (2.15) and $|y|\le\sqrt{2h} = \sqrt{2J_0^{-1}(I)}$.

For $i=1$, we have

$$\frac{\partial^{1+\ell}}{\partial\theta\partial t^\ell}H_1(\theta,I,t) = -T(h)g(x)E^{(\ell)}(t).$$

[2]The first inequality of (2.15) is the conclusion of [5, Lemma A4.1] and the second one in (2.15) can be found in the proof of that lemma.

By (1.4), (1.7) and (2.15), it follows that

$$\begin{aligned}\left|\frac{\partial^k}{\partial I^k} g(x)\right| &\leq \sum_{k_1+\cdots+k_s=k}\left|G^{(s+1)}(x)\cdot\frac{\partial^{k_1}x}{\partial I^{k_1}}\cdots\frac{\partial^{k_s}x}{\partial I^{k_s}}\right| \\ &\leq C\cdot\sum\left|G^{(s+1)}(x)\cdot I^{-k_1}|x|\cdots I^{-k_s}|x|\right| \\ &\leq C\cdot I^{-k}\cdot\sum|x^sG^{(s+1)}(x)|\leq C\cdot I^{-k}\cdot\left|\frac{G(x)}{x}\right| \\ &\leq C\cdot I^{-k}\cdot|g(x)|\leq C\cdot I^{-k}\cdot\max\left(g(x_+),|g(x_-)|\right) \\ &\leq C\cdot I^{-k}\cdot\frac{h}{\min\{x_+,|x_-|\}}.\end{aligned}$$

Hence from Lemma 2.1, (2.3) and (2.16), we have

$$\begin{aligned}\left|\frac{\partial^{k+i+\ell}}{\partial I^k\partial\theta^i\partial t^\ell}H_1(\theta,I,t)\right| &\leq \left|\sum_{m+n=k}\frac{d^m}{dI^m}T(h)\cdot\frac{\partial^n}{\partial I^n}g(x)\cdot E^{(\ell)}(t)\right| \\ &\leq C\cdot I^{-k}T(h)\cdot\frac{h}{\min\{x_+,|x_-|\}} \\ &\leq C\cdot I^{-k}\cdot\frac{J_0(h)}{\min\{x_+,|x_-|\}} \\ &\leq C\cdot I^{-k}\cdot\sqrt{h}\cdot\left(1+\frac{\max\{x_+,|x_-|\}}{\min\{x_+,|x_-|\}}\right) \\ &\leq C\cdot I^{-k}\cdot\sqrt{h}=C\cdot I^{-k}\cdot\sqrt{J_0^{-1}(I)}.\end{aligned}$$

This completes the proof. □

3. New action and angle variables

Now we are concerned with the Hamiltonian system (2.6) with Hamiltonian function $\varrho(\theta, I, t)$ given by (2.7). Note that

$$Id\theta-\varrho dt=-(\varrho dt-Id\theta).$$

This means that if one can solve $I=\mathcal{I}(t,\varrho,\theta)$ from (2.7) (t and θ as parameters) as a function of t,ϱ and θ, then

$$\frac{d\varrho}{d\theta}=-\frac{\partial\mathcal{I}}{\partial t}(t,\varrho,\theta),\quad \frac{dt}{d\theta}=\frac{\partial\mathcal{I}}{\partial\varrho}(t,\varrho,\theta), \tag{3.1}$$

i.e., (3.1) is also a Hamiltonian system with Hamiltonian function $\mathcal{I}(t,\varrho,\theta)$ and now the action, angle and time variables are ϱ, t and θ, respectively. This trick has been used in [4] and [5].

From Lemmas 2.1 and 2.3, we know that

$$\frac{\partial\varrho}{\partial I}(\theta,I,t)\neq 0,$$

for $I\gg 1$. Hence by the implicit function theorem, there is a function $\mathcal{I}(t,\varrho,\theta)$ such that

$$\varrho(\theta,\mathcal{I}(t,\varrho,\theta),t)=\varrho. \tag{3.2}$$

The aim of this section is to obtain an explicit form of $\mathcal{I}$. Because

$$0 \leq \frac{|H_1(\theta, I, t)|}{J_0^{-1}(I)} \leq C \cdot \frac{1}{\sqrt{J_0^{-1}(I)}} \to 0, \quad \text{as } I \to \infty,$$

there is a function $R(t, \varrho, \theta)$ with $|R| \leq \frac{1}{2}\varrho$ such that

$$\mathcal{I}(t, \varrho, \theta) = J_0(\varrho - R(t, \varrho, \theta)), \tag{3.3}$$

for $\varrho \gg 1$. Let

$$J_1(t, \varrho, \theta) = \int_0^1 J_0'(\varrho - sR(t, \varrho, \theta)) \cdot R(t, \varrho, \theta) ds.$$

Then

$$\mathcal{I}(t, \varrho, \theta) = J_0(\varrho) + J_1(t, \varrho, \theta). \tag{3.4}$$

Lemma 3.1. *The perturbation term $J_1(t, \varrho, \theta)$ possesses the estimates*

$$\left| \frac{\partial^{k+\ell+i}}{\partial \varrho^k \partial t^\ell \partial \theta^i} J_1(t, \varrho, \theta) \right| \leq C \cdot \varrho^{-k-1/2} \cdot J_0(\varrho), \tag{3.5}$$

for $k + i + \ell \leq 6$, $i = 0, 1$ and $\varrho \gg 1$.

In the proof of this statement, we need the following technical lemma about the estimates of $R(t, \varrho, \theta)$.

Lemma 3.2. *The function $R(t, \varrho, \theta)$ possesses the following estimates:*

$$\left| \frac{\partial^{k+\ell+i}}{\partial \varrho^k \partial t^\ell \partial \theta^i} R(t, \varrho, \theta) \right| \leq C \cdot \varrho^{-k+1/2}, \tag{3.6}$$

for $k + i + \ell \leq 7$, $i = 0, 1$, $\ell \leq 6$ and $\varrho \gg 1$.

We first give a proof of Lemma 3.1, and then show the estimates of $R(t, \varrho, \theta)$ in (3.6) are true.

Proof of Lemma 3.1. From the definition of J_1, we have

$$\frac{\partial^{k+\ell+i}}{\partial \varrho^k \partial t^\ell \partial \theta^i} J_1(t, \varrho, \theta) = \sum \int_0^1 \frac{\partial^{k_1+\ell_1+i_1}}{\partial \varrho^{k_1} \partial t^{\ell_1} \partial \theta^{i_1}} J_0'(\varrho - sR) \cdot \frac{\partial^{k_2+\ell_2+i_2}}{\partial \varrho^{k_2} \partial t^{\ell_2} \partial \theta^{i_2}} R ds,$$

where $k_1 + k_2 = k$, $\ell_1 + \ell_2 = \ell$ and $i_1 + i_2 = i$. Now we estimate the first factor of the integrand.

When $i_1 = 0$, one may prove that the following equality holds

$$\begin{aligned} \frac{\partial^{m+n}}{\partial \varrho^m \partial t^n} J_0'(\varrho - sR) \quad &= \quad \sum J_0^{(p+q+1)}(\varrho - sR) \cdot \frac{\partial^{m_1} u}{\partial \varrho^{m_1}} \cdots \frac{\partial^{m_p} u}{\partial \varrho^{m_p}} \\ &\quad \cdot \frac{\partial^{j_1+n_1} u}{\partial \varrho^{j_1} \partial t^{n_1}} \cdots \frac{\partial^{j_q+n_q} u}{\partial \varrho^{j_q} \partial t^{n_q}}, \end{aligned}$$

where $u := \varrho - sR$, $p \leq m$, $q \leq n$, $n_1, \cdots, n_q > 0$, $m_1, \cdots, m_p > 0$ and $n_1 + \cdots + n_q = n$, $m_1 + \cdots + m_p + j_1 + \cdots + j_q = m$. Assume that there are $\alpha (\leq p)$ members in $\{m_1, \cdots, m_p\}$ which equal to 1. Notice that if $\varrho \gg 1$, then $|\partial u / \partial \varrho| \leq C$

$|\partial^k u/\partial \varrho^k| \le |\partial^k R/\partial \varrho^k|$ for $k > 1$ and $|\partial^{k+\ell} u/\partial \varrho^k \partial t^\ell| \le |\partial^{k+\ell} R/\partial \varrho^k \partial t^\ell|$ if $\ell > 0$. So from Lemmas 2.1 and 3.2, we have

$$\begin{aligned}\left|\frac{\partial^{m+n}}{\partial \varrho^m \partial t^n} J_0'(\varrho - sR)\right| &\le C \cdot \frac{J_0(\varrho - sR)}{(\varrho - sR)^{p+q+1}} \cdot \varrho^{-(m_1+\cdots+m_p-\alpha)+(p-\alpha)/2} \\ &\quad \cdot \varrho^{-(j_1+\cdots+j_q)+q/2} \\ &\le C \cdot \frac{1}{(\frac{1}{2}\varrho)^{p+q+1}} J_0(\frac{3}{2}\varrho) \cdot \varrho^{-m+\frac{p+q+\alpha}{2}} \\ &\le C \cdot \varrho^{-m-1} \cdot J_0(\varrho) \cdot \varrho^{-\frac{p+q-\alpha}{2}} \\ &\le C \cdot \varrho^{-m-1} \cdot J_0(\varrho),\end{aligned}$$

where we have used the following inequalities

$$0 \le s \le 1, \quad |R| \le \frac{1}{2}\varrho, \quad J_0(\frac{3}{2}\varrho) \le C \cdot J_0(\varrho).$$

Similarly, one can get the same estimate for $i_1 = 1$ as well as one for $i_1 = 0$. Hence

$$\left|\frac{\partial^{k_1+\ell_1+i_1}}{\partial \varrho^{k_1} \partial t^{\ell_1} \partial \theta^{i_1}} J_0'(\varrho - sR)\right| \le C \cdot \varrho^{-k_1-1} \cdot J_0(\varrho),$$

which yields that

$$\begin{aligned}\left|\frac{\partial^{k+\ell+i}}{\partial \varrho^k \partial t^\ell \partial \theta^i} J_1(\varrho - sR)\right| &\le C \cdot \varrho^{-k_1-1} \cdot J_0(\varrho) \cdot \varrho^{-k_2+1/2} \\ &\le C \cdot \varrho^{-k-1/2} \cdot J_0(\varrho).\end{aligned}$$

□

Proof of Lemma 3.2. From (2.7), (3.2) and (3.3), it follows that

$$R(t, \varrho, \theta) = H_1(\theta, J_0(\varrho - R), t). \tag{3.7}$$

Using Lemma 2.3, one has

$$|R| \le C \cdot \sqrt{J_0^{-1}(J_0(\varrho - R))} = C \cdot \sqrt{\varrho - R} \le C \cdot \sqrt{\varrho}.$$

From Lemma 2.1, it is easy to see that J_0 is increasing and

$$J_0(\frac{1}{2}\varrho) \le J_0(\varrho) \le J_0(\frac{3}{2}\varrho) \le C \cdot J_0(\frac{1}{2}\varrho). \tag{3.8}$$

Indeed, the first two inequalities hold because of $J_0'(\varrho) > 0$ by (2.9). On the other hand, from (2.10) and (2.12), it follows that J_0' is increasing and

$$J_0'(h) \cdot h \le (\frac{3}{2} - \sigma_1) J_0(h).$$

So we obtain

$$J_0(\frac{3}{2}\varrho) - J_0(\frac{1}{2}\varrho) = J_0'(\xi)\varrho \le \frac{2}{3} J_0'(\frac{3}{2}\varrho) \cdot \frac{3}{2}\varrho \le J_0(\frac{3}{2}\varrho) - \frac{2}{3}\sigma_1 \cdot J_0(\frac{3}{2}\varrho),$$

which yields the third inequality.

By Lemmas 2.1, 2.3 and (3.8), we have, for $\varrho \gg 1$,

$$\begin{aligned}
&\left|\frac{\partial H_1}{\partial I}(\theta, J_0(\varrho - R), t) \cdot J_0'(\varrho - R)\right| \\
&\qquad \le C \cdot \frac{1}{J_0(\varrho - R)} \cdot \sqrt{J_0^{-1}(J_0(\varrho - R))} \cdot J_0'(\varrho - R) \\
&\qquad \le C \cdot \frac{1}{J_0(\varrho/2)} \cdot \sqrt{\frac{3}{2}\varrho} \cdot J_0'(\frac{3}{2}\varrho) \\
&\qquad \le C \cdot \frac{1}{J_0(\frac{1}{2}\varrho)} \cdot \sqrt{\frac{3}{2}\varrho} \cdot \frac{1}{\frac{3}{2}\varrho} \cdot J_0(\frac{3}{2}\varrho) \\
&\qquad \le C \cdot \varrho^{-1/2} \le 1/2.
\end{aligned}$$

Let

$$\Delta := 1 + \frac{\partial H_1}{\partial I}(\theta, J_0(\varrho - R), t) \cdot J_0'(\varrho - R).$$

Then $\Delta \ge 1/2$. Differentiating on both sides of (3.7) with respect to ϱ, t and θ, respectively, we get

$$\begin{aligned}
\Delta \cdot \frac{\partial R}{\partial \varrho}(t, \varrho, \theta) &= \frac{\partial H_1}{\partial I}(\theta, J_0(\varrho - R), t) \cdot J_0'(\varrho - R) := g_1(t, \varrho, \theta), \\
\Delta \cdot \frac{\partial R}{\partial t}(t, \varrho, \theta) &= \frac{\partial H_1}{\partial t}(\theta, J_0(\varrho - R), t) := g_2(t, \varrho, \theta), \\
\Delta \cdot \frac{\partial R}{\partial \theta}(t, \varrho, \theta) &= \frac{\partial H_1}{\partial \theta}(\theta, J_0(\varrho - R), t) := g_3(t, \varrho, \theta).
\end{aligned} \tag{3.9}$$

By a direct computation, one may get

$$|g_1(t, \varrho, \theta)| \le C \cdot \varrho^{-1/2}, \quad |g_2(t, \varrho, \theta)| \le C \cdot \varrho^{1/2}, \quad |g_3(t, \varrho, \theta)| \le C \cdot \varrho^{1/2}.$$

Hence, by (3.9), we have

$$\left|\frac{\partial^{k+i+\ell}}{\partial \varrho^k \partial \theta^i \partial t^\ell} R(t, \varrho, \theta)\right| \le C \cdot \varrho^{-k+1/2}$$

for $k + i + \ell = 1$.

The conclusion of Lemma 3.2 has already been verified when $k + i + \ell = 0$ and 1. For the general case, we need the following claim.

Claim. If

$$\left|\frac{\partial^{k+\ell}}{\partial \varrho^k \partial t^\ell} R(t, \varrho, \theta)\right| \le C \cdot \varrho^{-k+1/2}$$

for all $k + \ell \le L_0$. Then for $k + \ell \le L_0$,

$$\begin{aligned}
\left|\frac{\partial^{k+\ell}}{\partial \varrho^k \partial t^\ell} \Delta(t, \varrho, \theta)\right| &\le C \cdot \varrho^{-k-1/2}, \quad (k + \ell > 0), \\
\left|\frac{\partial^{k+\ell}}{\partial \varrho^k \partial t^\ell} g_1(t, \varrho, \theta)\right| &\le C \cdot \varrho^{-k-1/2}, \\
\left|\frac{\partial^{k+\ell}}{\partial \varrho^k \partial t^\ell} g_2(t, \varrho, \theta)\right|, \qquad \left|\frac{\partial^{k+\ell}}{\partial \varrho^k \partial t^\ell} g_3(t, \varrho, \theta)\right| &\le C \cdot \varrho^{-k+1/2}.
\end{aligned}$$

Before proving this claim, let us show the statement of Lemma 3.2.

By induction, we assume that

$$\left|\frac{\partial^{k+\ell}}{\partial\varrho^k\partial t^\ell}R(t,\varrho,\theta)\right| \le C\cdot\varrho^{-k+1/2}$$

for all $k+\ell\le L_0$. Then applying $\frac{\partial^{k+\ell}}{\partial\varrho^k\partial t^\ell}$ to both sides of the first equality in (3.9), one has

$$\sum\left(\frac{\partial^{m+n}}{\partial\varrho^m\partial t^n}\Delta\right)\cdot\left(\frac{\partial^{m'+n'}}{\partial\varrho^{m'}\partial t^{n'}}R\right)+\Delta\cdot\left(\frac{\partial^{k+1+\ell}}{\partial\varrho^{k+1}\partial t^\ell}R\right)=\frac{\partial^{k+\ell}}{\partial\varrho^k\partial t^\ell}g_1,$$

where $m+m'=k+1$, $n+n'=\ell$ and $m+n>0$, $m\le k$. So by the hypothesis of induction,

$$\begin{aligned}\left|\frac{\partial^{k+1+\ell}}{\partial\varrho^{k+1}\partial t^\ell}R\right| &\le 2\Delta\cdot\left|\frac{\partial^{k+1+\ell}}{\partial\varrho^{k+1}\partial t^\ell}R\right|\\ &\le 2\sum\left|\frac{\partial^{m+n}}{\partial\varrho^m\partial t^n}\Delta\right|\cdot\left|\frac{\partial^{m'+n'}}{\partial\varrho^{m'}\partial t^{n'}}R\right|+2\left|\frac{\partial^{k+\ell}}{\partial\varrho^k\partial t^\ell}g_1\right|\\ &\le C\cdot\varrho^{-m-1/2-m'+1/2}+C\cdot\varrho^{-k-1/2}\\ &= C\cdot\varrho^{-(k+1)}+C\cdot\varrho^{-(k+1)+1/2}\\ &\le C\cdot\varrho^{-(k+1)+1/2}.\end{aligned}$$

Similarly, applying $\frac{\partial^{k+\ell}}{\partial\varrho^k\partial t^\ell}$ to both sides of the second and third equality of (3.9) yields that

$$\left|\frac{\partial^{k+1+\ell}}{\partial\varrho^k\partial t^{\ell+1}}R\right|,\quad \left|\frac{\partial^{k+1+\ell}}{\partial\varrho^k\partial\theta\partial t^\ell}R\right|\le C\cdot\varrho^{-k+1/2}.$$

That is, if the conclusion is true for $k+i+\ell\le L_0$, then it is also valid for $k+i+\ell\le L_0+1$.

To finish the proof, it suffices to see that the statements of the claim are true.

Proof of Claim. We first give a proof of the estimate of the function g_1.

Let

$$f_1(t,\varrho,\theta)=J_0'(\varrho-R)$$

and

$$f_2(t,\varrho,\theta)=\frac{\partial H_1}{\partial I}(\theta,J_0(\varrho-R),t).$$

Then if

$$\left|\frac{\partial^{k+\ell}}{\partial\varrho^k\partial t^\ell}R\right|\le C\cdot\varrho^{-k+1/2},\quad \text{for } k+\ell\le L_0, \tag{3.10}$$

we have

$$\left|\frac{\partial^{k+\ell}}{\partial\varrho^k\partial t^\ell}f_1\right|\le C\cdot\varrho^{-k-1}J_0(\varrho),\quad \text{for } k+\ell\le L_0, \tag{3.11}$$

$$\left|\frac{\partial^{k+\ell}}{\partial\varrho^k\partial t^\ell}f_2\right|\le C\cdot\varrho^{-k+1/2}\cdot\frac{1}{J_0(\varrho)},\quad \text{for } k+\ell\le L_0. \tag{3.12}$$

Proof of (3.11). (i) When $k+\ell=0$, we have

$$\begin{aligned}|f_1|=|J_0'(\varrho-R)| &\leq C\cdot(\varrho-R)^{-1}J_0(\varrho-R)\\ &\leq C\cdot(\frac{1}{2}\varrho)^{-1}J_0(\frac{3}{2}\varrho)\leq C\cdot\varrho^{-1}J_0(\varrho),\end{aligned}$$

where we have used the inequalities

$$|R(t,\varrho,\theta)|\leq\frac{1}{2}\varrho,\quad J_0(\frac{3}{2}\varrho)\leq C\cdot J_0(\varrho).$$

(ii) When $k>0$, $\ell=0$, the following equality holds

$$\frac{\partial^k}{\partial\varrho^k}f_1=\sum J_0^{(q+1)}(\varrho-R)\cdot\frac{\partial^{j_1}u}{\partial\varrho^{j_1}}\cdots\frac{\partial^{j_q}u}{\partial\varrho^{j_q}},$$

where $0<q\leq k$, $j_1,\cdots,j_q>0$, $j_1+\cdots+j_q=k$ and $u:=\varrho-R$. Assume that there are $b(\leq q)$ members in $\{j_1,\cdots,j_q\}$ which are equal to 1. Then

$$\begin{aligned}\left|\frac{\partial^k}{\partial\varrho^k}f_1\right| &\leq C\cdot(\varrho-R)^{-q-1}J_0(\varrho-R)\cdot\varrho^{-(j_1+\cdots+j_q-b)+\frac{1}{2}(q-b)}\\ &\leq C\cdot\varrho^{-q-1}\cdot J_0(\frac{3}{2}\varrho)\cdot\varrho^{-k+\frac{1}{2}(q+b)}\\ &\leq C\cdot\varrho^{-k-1}J_0(\varrho)\cdot\varrho^{-\frac{1}{2}(q-b)}\\ &\leq C\cdot\varrho^{-k-1}J_0(\varrho).\end{aligned}$$

(iii) When $k=0,\ell>0$, we have from the equality

$$\frac{\partial^\ell}{\partial t^\ell}f_1=\sum J_0^{(p+1)}(\varrho-R)\cdot\frac{\partial^{\ell_1}u}{\partial t^{\ell_1}}\cdots\frac{\partial^{\ell_p}u}{\partial t^{\ell_p}},$$

where $p\leq\ell$, $\ell_1,\cdots,\ell_p>0$ and $\ell_1+\cdots+\ell_p=\ell$, it follows that

$$\begin{aligned}\left|\frac{\partial^\ell}{\partial t^\ell}f_1\right| &\leq C\cdot(\varrho-R)^{-p-1}J_0(\varrho-R)\cdot(\varrho^{1/2})^p\\ &\leq C\cdot\varrho^{-p-1}\cdot J_0(\varrho)\cdot\varrho^{p/2}\\ &\leq C\cdot\varrho^{-1}J_0(\varrho).\end{aligned}$$

(iv) When $k,\ell>0$, it is easy to see that

$$\begin{aligned}\frac{\partial^{k+\ell}}{\partial\varrho^k\partial t^\ell}f_1(t,\varrho,\theta) &= \sum J_0^{(p+q+1)}(\varrho-R)\cdot\frac{\partial^{j_1}u}{\partial\varrho^{j_1}}\cdots\frac{\partial^{j_q}u}{\partial\varrho^{j_q}}\\ &\quad\cdot\frac{\partial^{\ell_1+k_1}u}{\partial\varrho^{k_1}\partial t^{\ell_1}}\cdots\frac{\partial^{\ell_p+k_p}u}{\partial\varrho^{k_p}\partial t^{\ell_p}},\end{aligned}$$

where $u=\varrho-R$ and

$$0\leq p\leq\ell,\ 0\leq q\leq k,\ j_1,\cdots,j_q,\ell_1,\cdots,\ell_p>0,\ \ k_1,\cdots,k_p\geq 0,$$
$$j_1+\cdots+j_q+k_1+\cdots+k_p=k,\ \ \ell_1+\cdots+\ell_p=\ell.$$

Assume that there are $b'(\leq q)$ members in $\{j_1,\cdots,j_q\}$ which are equal to 1. Then

$$\begin{aligned}\left|\frac{\partial^{k+\ell}}{\partial\varrho^k\partial t^\ell}f_1\right| &\leq C\cdot(\varrho-R)^{-(p+q+1)}J_0(\varrho-R)\cdot\varrho^{-(j_1+\cdots+j_q-b')}\cdot(\varrho^{1/2})^{q-b'}\\ &\quad\cdot\varrho^{-(k_1+\cdots+k_p)}\cdot\varrho^{p/2}\\ &\leq C\cdot\varrho^{-k-1}\cdot J_0(\varrho)\cdot\varrho^{-\frac{p+q-b'}{2}}\\ &\leq C\cdot\varrho^{-k-1}J_0(\varrho).\end{aligned}$$

The proof of (3.11) is completed.

From (3.11), it is not difficult to prove that if (3.10) holds, then

$$\left|\frac{\partial^{k+\ell}}{\partial \varrho^k \partial t^\ell} J_0(\varrho - R)\right| \le C \cdot \varrho^{-k} \cdot J_0(\varrho), \quad \text{for } k+\ell \le L_0. \tag{3.13}$$

Using this estimate and (3.11), one can verify (3.12).

Proof of (3.12). (i) When $k+\ell = 0$, from Lemma 2.3 it follows that

$$\begin{aligned} |f_2| = \left|\frac{\partial H_1}{\partial I}(\theta, J_0(\varrho - R), t)\right| &\le C \cdot \frac{1}{J_0(\varrho - R)} \cdot \sqrt{J_0^{-1}(J_0(\varrho - R))} \\ &\le C \cdot \frac{\sqrt{\varrho - R}}{J_0(\varrho - R)} \le C \cdot \frac{\sqrt{\frac{3}{2}\varrho}}{J_0(\frac{1}{2}\varrho)} \\ &\le C \cdot \varrho^{1/2} \frac{1}{J_0(\varrho)}, \end{aligned}$$

where we have used the inequalities

$$|R(t, \varrho, \theta)| \le \frac{1}{2}\varrho, \quad J_0(\frac{1}{2}\varrho) \ge c \cdot J_0(\varrho).$$

(ii) When $k > 0$, $\ell = 0$, the following equality holds

$$\frac{\partial^k}{\partial \varrho^k} f_2 = \sum \frac{\partial^{q+1} H_1}{\partial I^{q+1}}(\theta, J_0(\varrho - R), t) \cdot \frac{\partial^{j_1}}{\partial \varrho^{j_1}} J_0(\varrho - R) \cdots \frac{\partial^{j_q}}{\partial \varrho^{j_q}} J_0(\varrho - R),$$

where $0 < q \le k$, $j_1, \cdots, j_q > 0$, $j_1 + \cdots + j_q = k$. By (3.13) and Lemma 2.3, we have

$$\begin{aligned} \left|\frac{\partial^k}{\partial \varrho^k} f_2\right| &\le C \cdot \sum \frac{1}{(J_0(\varrho - R))^{q+1}} \cdot \sqrt{J_0^{-1}(J_0(\varrho - R))} \cdot \varrho^{-(j_1 + \cdots + j_q)} (J_0(\varrho))^q \\ &\le C \cdot \frac{\varrho^{-k}}{J_0(\varrho)} \cdot \sqrt{\varrho - R} \\ &\le C \cdot \varrho^{-k+1/2} \cdot \frac{1}{J_0(\varrho)}. \end{aligned}$$

(iii) When $k = 0, \ell > 0$, from Lemma 2.3, (3.13) and the equality

$$\frac{\partial^\ell}{\partial t^\ell} f_2 = \sum \frac{\partial^{\alpha+\beta+1} H_1}{\partial I^{\alpha+1} \partial t^\beta}(\theta, J_0(\varrho - R), t) \cdot \frac{\partial^{\ell_1}}{\partial t^{\ell_1}} J_0(\varrho - R) \cdots \frac{\partial^{\ell_\alpha}}{\partial t^{\ell_\alpha}} J_0(\varrho - R),$$

where $\alpha \le \ell$ and $\ell_1 + \cdots + \ell_\alpha + \beta = \ell$, it follows that

$$\begin{aligned} \left|\frac{\partial^\ell}{\partial t^\ell} f_2\right| &\le C \cdot \sum \frac{1}{(J_0(\varrho - R))^{\alpha+1}} \cdot \sqrt{J_0^{-1}(J_0(\varrho - R))} \cdot (J_0(\varrho))^\alpha \\ &\le C \cdot \frac{1}{J_0(\varrho)} \cdot \sqrt{\varrho - R} \le C \cdot \varrho^{1/2} \cdot \frac{1}{J_0(\varrho)}. \end{aligned}$$

(iv) When $k, \ell > 0$, by a direct computation, one may obtain

$$\begin{aligned}\frac{\partial^{k+\ell}}{\partial \varrho^k \partial t^\ell} f_2(t, \varrho, \theta) &= \sum \frac{\partial^{\alpha+\beta+\gamma+1} H_1}{\partial I^{\alpha+\beta+1} \partial t^\gamma}(\theta, J_0(\varrho - R), t) \\ &\quad \cdot \frac{\partial^{j_1}}{\partial \varrho^{j_1}} J_0(\varrho - R) \cdots \frac{\partial^{j_\beta}}{\partial \varrho^{j_\beta}} J_0(\varrho - R) \\ &\quad \cdot \frac{\partial^{k_1+\ell_1}}{\partial \varrho^{k_1} \partial t^{\ell_1}} J_0(\varrho - R) \cdots \frac{\partial^{k_\alpha+\ell_\alpha}}{\partial \varrho^{k_\alpha} \partial t^{\ell_\alpha}} J_0(\varrho - R),\end{aligned}$$

where

$$0 \le \beta \le k,\ 0 \le \alpha + \gamma \le \ell,\ j_1 + \cdots + j_\beta + k_1 + \cdots + k_\alpha = k,\ \ell_1 + \cdots + \ell_\alpha + \gamma = \ell.$$

Then

$$\begin{aligned}\left|\frac{\partial^{k+\ell}}{\partial \varrho^k \partial t^\ell} f_2\right| &\le C \cdot \frac{1}{(J_0(\varrho - R))^{\alpha+\beta+1}} \cdot \sqrt{J_0^{-1}(J_0(\varrho - R))} \cdot \varrho^{-j_1} J_0(\varrho) \cdots \varrho^{-j_\beta} J_0(\varrho) \\ &\quad \cdot \varrho^{-k_1} J_0(\varrho) \cdots \varrho^{-k_\alpha} J_0(\varrho) \\ &\le C \cdot \varrho^{-k} \cdot \sqrt{\varrho - R} \cdot \frac{1}{(J_0(\varrho/2))^{\alpha+\beta+1}} \cdot (J_0(\varrho))^{\alpha+\beta} \\ &\le C \cdot \varrho^{-k+1/2} \cdot \frac{1}{J_0(\varrho)}.\end{aligned}$$

From $g_1 = f_1 \cdot f_2$, it follows that

$$\begin{aligned}\left|\frac{\partial^{k+\ell}}{\partial \varrho^k \partial t^\ell} g_1\right| &= \left|\sum \frac{\partial^{k_1+\ell_1}}{\partial \varrho^{k_1} \partial t^{\ell_1}} f_1 \cdot \frac{\partial^{k_2+\ell_2}}{\partial \varrho^{k_2} \partial t^{\ell_2}} f_2\right| \\ &\le C \cdot \varrho^{-k_1-1} J_0(\varrho) \cdot \varrho^{-k_2+1/2} \cdot \frac{1}{J_0(\varrho)} \\ &= C \cdot \varrho^{-k-1/2}.\end{aligned}$$

Note that if $k + \ell > 0$,

$$\left|\frac{\partial^{k+\ell}}{\partial \varrho^k \partial t^\ell} \Delta\right| = \left|\frac{\partial^{k+\ell}}{\partial \varrho^k \partial t^\ell} g_1\right| \le C \cdot \varrho^{-k-1/2}.$$

The proofs of the estimates for g_2 and g_3 are very similar to the proof of (3.12), so we omit it.

4. Proof of Theorem 1

Up to now, we have given an equivalent form of Eq. (1.1), that is, the system (3.1), which is expressed in the action and angle variables (ϱ, t). However, its Poincaré mapping is far from a small perturbation of the standard twist mapping $(t, \varrho) \mapsto (t + \varrho, \varrho)$. Hence, one cannot using Moser's twist theorem directly. In this section, we first introduce some transformations such that in the transformed system, the terms depending on the new angle variable are very small if the new action variable is sufficiently large and then prove, based on Moser's twist theorem, the statement of Theorem 1.

Lemma 4.1. *There is a canonical transformation $\Psi : (\lambda, \tau) \mapsto (\varrho, t)$ of the form*

$$\Psi : \varrho = \lambda + U(\tau, \lambda, \theta), \quad t = \tau + V(\tau, \lambda, \theta), \tag{4.1}$$

where the functions U and V are 1-periodic in θ and satisfy

$$U(\tau, \lambda, \theta)/\lambda, \quad V(\tau, \lambda, \theta) \to 0 \quad \text{as } \lambda \to \infty$$

uniformly for $(\tau, \theta) \in T^2$ such that under this mapping, the system (3.1) with the Hamiltonian function $\mathcal{I}$ in (3.4) is changed into the form

$$\frac{d\lambda}{d\theta} = -\frac{\partial \mathcal{K}}{\partial \tau}(\tau, \lambda, \theta), \quad \frac{d\tau}{d\theta} = \frac{\partial \mathcal{K}}{\partial \lambda}(\tau, \lambda, \theta), \tag{4.2}$$

where

$$\mathcal{K}(\tau, \lambda, \theta) = J_0(\lambda) + [J_1](\lambda, \theta) + \mathcal{K}_1(\tau, \lambda, \theta) \tag{4.3}$$

with

$$[J_1](\lambda, \theta) = \int_0^1 J_1(t, \lambda, \theta) dt.$$

Moreover, the new perturbation $\mathcal{K}_1$ possesses the estimate:

$$\left| \frac{\partial^{k+\ell}}{\partial \lambda^k \partial \tau^\ell} \mathcal{K}_1(\tau, \lambda, \theta) \right| \leq C \cdot \lambda^{-k+1/2} \tag{4.4}$$

for $k + \ell \leq 5$.

Proof. We will look for the required transformation Ψ determined by a generating function $\mathcal{F}(t, \lambda, \theta)$ in the following way:

$$\varrho = \lambda + \frac{\partial \mathcal{F}}{\partial t}(t, \lambda, \theta), \quad \tau = t + \frac{\partial \mathcal{F}}{\partial \lambda}(t, \lambda, \theta), \tag{4.5}$$

where the function $\mathcal{F}$ will be given later. Under Ψ, the transformed system of (3.1) is of the form

$$\frac{d\lambda}{d\theta} = -\frac{\partial \mathcal{K}}{\partial \tau}(\tau, \lambda, \theta), \quad \frac{d\tau}{d\theta} = \frac{\partial \mathcal{K}}{\partial \lambda}(\tau, \lambda, \theta),$$

where

$$\mathcal{K}(\tau, \lambda, \theta) = J_0(\lambda + \frac{\partial \mathcal{F}}{\partial t}) + J_1(t, \lambda + \frac{\partial \mathcal{F}}{\partial t}, \theta) + \frac{\partial \mathcal{F}}{\partial \theta}.$$

By Taylor's formula, one can write

$$\mathcal{K}(\tau, \lambda, \theta) = J_0(\lambda) + J_0'(\lambda) \cdot \frac{\partial \mathcal{F}}{\partial t} + J_1(t, \lambda, \theta) + \mathcal{K}_1(\tau, \lambda, \theta),$$

where

$$\begin{aligned} \mathcal{K}_1(\tau, \lambda, \theta) = \frac{\partial \mathcal{F}}{\partial \theta} &+ \int_0^1 (1-s) J_0''(\lambda + s\frac{\partial \mathcal{F}}{\partial t}) \cdot \left(\frac{\partial \mathcal{F}}{\partial t}\right)^2 ds \\ &+ \int_0^1 \frac{\partial J_1}{\partial \varrho}(t, \lambda + s\frac{\partial \mathcal{F}}{\partial t}, \theta) \cdot \frac{\partial \mathcal{F}}{\partial t} ds. \end{aligned}$$

We now choose $\mathcal{F}$:

$$\mathcal{F}(t, \lambda, \theta) = -\int_0^t \frac{1}{J_0'(\lambda)} \cdot (J_1(t, \lambda, \theta) - [J_1](\lambda, \theta)) \, dt.$$

Then $\mathcal{K}$ is of the form (4.3).

To complete the proof, it suffices to show that $\mathcal{K}_1$ satisfies (4.4).

From (2.9) and (3.5), it follows that

$$\left| \frac{\partial^{k+\ell+i}}{\partial \lambda^k \partial t^\ell \partial \theta^i} \mathcal{F}(t, \lambda, \theta) \right| \leq C \cdot \lambda^{-k+1/2} \tag{4.6}$$

for $k+i+\ell \leq 6$ and $i=0,1$. In particular,

$$\left|\frac{\partial^2}{\partial\lambda\partial t}\mathcal{F}(t,\lambda,\theta)\right| \leq C\cdot\lambda^{-1/2} \leq 1/2$$

if $\lambda \gg 1$. So one can solve the second equation of (4.5) for t,

$$t = \tau + V(\tau,\lambda,\theta)$$

where the function V satisfies

$$V(\tau,\lambda,\theta) = -\frac{\partial\mathcal{F}}{\partial\lambda}(\tau+V,\lambda,\theta).$$

Set

$$U(\tau,\lambda,\theta) = \frac{\partial\mathcal{F}}{\partial t}(\tau+V,\lambda,\theta).$$

Then the canonical transformation Ψ is of the form (4.1). Moreover, similar to the proof of [2, Lemma 2], one can verify that

(4.7)
$$\left|\frac{\partial^{k+\ell}}{\partial\lambda^k\partial\tau^\ell}U(\tau,\lambda,\theta)\right| \leq C\cdot\lambda^{-k+1/2}, \quad \left|\frac{\partial^{k+\ell}}{\partial\lambda^k\partial\tau^\ell}V(\tau,\lambda,\theta)\right| \leq C\cdot\lambda^{-k-1/2},$$

for $k+\ell\leq 5$, and U/λ, $V\to 0$ as $\lambda\to+\infty$.

Let

$$\begin{aligned}
\phi_1(\tau,\lambda,\theta) &= \frac{\partial\mathcal{F}}{\partial\theta}(\tau+V,\lambda,\theta),\\
\phi_2(\tau,\lambda,\theta) &= \int_0^1(1-s)J_0''(\lambda+sU)\cdot U^2ds,\\
\phi_3(\tau,\lambda,\theta) &= \int_0^1\frac{\partial J_1}{\partial\varrho}(\tau+V,\lambda+sU,\theta)\cdot Uds.
\end{aligned}$$

It is not difficult to prove that[3]

$$\begin{aligned}
\left|\frac{\partial^{k+\ell}}{\partial\lambda^k\partial\tau^\ell}\phi_1(\tau,\lambda,\theta)\right| &\leq C\cdot\lambda^{-k+1/2},\\
\left|\frac{\partial^{k+\ell}}{\partial\lambda^k\partial\tau^\ell}\phi_2(\tau,\lambda,\theta)\right| &\leq C\cdot\lambda^{-k-1}J_0(\lambda),\\
\left|\frac{\partial^{k+\ell}}{\partial\lambda^k\partial\tau^\ell}\phi_3(\tau,\lambda,\theta)\right| &\leq C\cdot\lambda^{-k-1}J_0(\lambda),
\end{aligned}$$

for $k+\ell\leq 5$. Note that by (2.13),

$$J_0(\lambda)\leq C\cdot\lambda^{3/2-\sigma_1}.$$

Hence we have

$$\left|\frac{\partial^{k+\ell}}{\partial\lambda^k\partial\tau^\ell}\mathcal{K}_1(\tau,\lambda,\theta)\right| \leq \sum_{i=1}^{3}\left|\frac{\partial^{k+\ell}}{\partial\lambda^k\partial\tau^\ell}\phi_i(\tau,\lambda,\theta)\right| \leq C\cdot\lambda^{-k+1/2}$$

for $k+\ell\leq 5$. The proof is finished. □

[3] The proof is very analogous to the proof of the claim stated in §3, so we omit it here.

For $\lambda_0 > 0$, denote by A_{λ_0} the annulus

$$A_{\lambda_0} := \{(\lambda, \tau, \theta) | \lambda \geq \lambda_0 \text{ and } (\tau, \theta) \in T^2\}.$$

In order to apply Moser's twist theorem, we need the following:

Lemma 4.2. *The Poincaré mapping $\mathcal{P}$ of (4.2) has the intersection property on A_{λ_0}, i.e., if Γ is an embedded circle in A_{λ_0} homotopic to a circle $\lambda = $ const. in A_{λ_0}, then $\mathcal{P}(\Gamma) \cap \Gamma \neq \emptyset$.*

The proof can be found in [2].

Under the diffeomorphism Ψ_1 on A_{λ_0} given by

$$\mu = J_0'(\lambda), \quad \tau = \tau, \quad \theta = \theta, \tag{4.8}$$

the transformed system of (4.2) is of the form

$$\frac{d\mu}{d\theta} = f_1(\tau, \mu, \theta), \quad \frac{d\tau}{d\theta} = \mu + f_2(\tau, \mu, \theta), \tag{4.9}$$

where

(4.10)

$$f_1(\tau, \mu, \theta) = -J_0''(\lambda) \cdot \frac{\partial \mathcal{K}_1}{\partial \tau}(\tau, \lambda, \theta), \quad f_2(\tau, \mu, \theta) = \frac{\partial [J_1]}{\partial \lambda}(\lambda, \theta) + \frac{\partial \mathcal{K}_1}{\partial \lambda}(\tau, \lambda, \theta),$$

with $\lambda = \lambda(\mu)$ defined by (4.8).

Now we estimate the functions f_1 and f_2.

By (2.13) and (3.5), we have

$$c \cdot \lambda^{\sigma_2} \leq J_0'(\lambda) \leq C \cdot \lambda^{\frac{1}{2} - \sigma_1}, \quad \left| \frac{\partial [J_1]}{\partial \lambda}(\lambda, \theta) \right| \leq C \cdot \lambda^{-\sigma_1}.$$

Hence

$$c \cdot \mu^{\frac{2}{1 - 2\sigma_1}} \leq |\lambda(\mu)| \leq C \cdot \mu^{1/\sigma_2}. \tag{4.11}$$

Obviously, $\lambda \gg 1 \Longleftrightarrow \mu \gg 1$. Moreover, by (2.10) and (2.14), we have

$$\left| \frac{\partial^k}{\partial \mu^k} \lambda(\mu) \right| \leq C \cdot \mu^{-k} \lambda(\mu). \tag{4.12}$$

From (2.14), (3.5), (4.4) and (4.12), it follows that, for $k + \ell \leq 4$,

$$\begin{aligned} \left| \frac{\partial^{k+\ell}}{\partial \mu^k \partial \tau^\ell} f_1(\tau, \mu, \theta) \right| &\leq C \cdot \mu^{-k} \cdot \lambda^{-2} J_0(\lambda) \cdot \lambda^{1/2} \leq C \cdot \lambda^{-3/2} J_0(\lambda) \\ &\leq C \cdot \lambda^{-\sigma_1} \leq C \cdot \mu^{-\frac{2\sigma_1}{1 - 2\sigma_1}}, \end{aligned}$$

and

$$\begin{aligned} \left| \frac{\partial^{k+\ell}}{\partial \mu^k \partial \tau^\ell} f_2(\tau, \mu, \theta) \right| &\leq C \cdot \mu^{-k} \cdot \left(\lambda^{-1/2} + \lambda^{-2} J_0(\lambda) \right) \\ &\leq C \cdot \left(\lambda^{-1/2} + \lambda^{-3/2} \cdot \lambda^{3/2 - \sigma_1} \right) \\ &\leq C \cdot \lambda^{-\sigma_1} \leq C \cdot \mu^{-\frac{2\sigma_1}{1 - 2\sigma_1}}. \end{aligned}$$

Now we are in a position to prove the statement of Theorem 1.

Proof of Theorem 1. Since the functions f_1 and f_2 are sufficiently small if $\mu \gg 1$, one can verify easily that the solutions of (4.9) do exist for $0 \le \theta \le 1$ if the intial value $\mu(0) = \mu$ sufficiently large. Integrating Eq. (4.9) from $\theta = 0$ to $\theta = 1$, we obtain that the Poincaré mapping Φ of (4.9) is of the form

$$\tau_1 = \tau_0 + \mu_0 + \Xi_1(\tau_0, \mu_0), \quad \mu_1 = \mu_0 + \Xi_2(\tau_0, \mu_0),$$

where Ξ_1 and Ξ_2 possess the estimates as well as f_1 and f_2, that is, for $k + \ell \le 4$,

$$\left| \frac{\partial^{k+\ell}}{\partial \mu_0^k \partial \tau_0^\ell} \Xi_1 \right|, \quad \left| \frac{\partial^{k+\ell}}{\partial \mu_0^k \partial \tau_0^\ell} \Xi_2 \right| \le C \cdot \mu_0^{-\delta_0},$$

where $\delta_0 = 2\sigma_1/(1 - 2\sigma_1) > 0$. Because Ψ_1 is a diffeomorphism, Φ possesses the intersection property on A_{μ_0}. Hence Φ satisfies all the assumptions of Moser's twist theorem [13], [15]. From this theorem, it follows that for any $\omega \gg 1$ satisfying

$$\left| \omega - \frac{p}{q} \right| \ge c_0 \cdot |q|^{-5/2}, \tag{4.13}$$

there is an invariant curve Γ of Φ and on which Φ is of the form

$$\tau_1 = \tau_0 + \omega.$$

One can conclude that there exist invariant curves of the Poincaré mapping of the system (2.1), which surrounding the origin $(x, y) = (0, 0)$ and arbitrarily far from the origin. So every solution of (2.1) is bounded. □

Applying Aubry-Mather theory, one can prove the following conclusions:

Theorem 2. *Under the conditions of Theorem 1, there is* $\varepsilon_0 > 0$ *such that*

(1) for any rational $p/q \in (0, \varepsilon_0)$*, Eq. (1.1) possesses an unlinked periodic solution (Birkhoff type) with period* q*;*

(2) for any irrational $\omega \in (0, \varepsilon_0)$*, Eq. (1.1) has generalized quasi-periodic solutions with frequency* $(1, \omega)$ *corresponding to the Mather set* $\mathcal{M}_\omega$*;*

(3) for any irrational $\omega \in (0, \varepsilon_0)$ *with* $1/\omega$ *satisfying (4.13), there is a quasi-periodic solution of (1.1) with frequency* $(1, \omega)$*.*

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